Correction : conjugué d'un nombre complexe

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Exercice 1

- $\overline{-i} = i$;
- $\overline{2+i} = 2-i$;
- $\overline{3-2i} = 3+2i$;
- $\overline{\left(\dfrac{i}{2}\right)} = -\dfrac{i}{2}$;
- $\overline{-3-i} = -3+i$;
- $\overline{-2i-5} = 2i-5$.

Exercice 2

1) $z = \dfrac{1}{i}$. Alors $\overline{z} = \dfrac{\overline{1}}{\overline{i}} = \dfrac{1}{-i} = \dfrac{i}{1} = \boxed{i}$.

2) $z = \dfrac{2i-1}{1-2i}$. Alors $\overline{z} = \dfrac{\overline{2i-1}}{\overline{1-2i}} = \dfrac{-2i-1}{1+2i} = \dfrac{(-2i-1)(1-2i)}{1^2+2^2} = \dfrac{-2i+4i^2-1+2i}{1+4} = \dfrac{-2i+4\times(-1)-1+2i}{5}$

$\overline{z} = \dfrac{-4-1+\cancel{2i}-\cancel{2i}}{5} = \dfrac{-5}{5} = \boxed{-1}$.

3) $z = (5+2i)^2$. Alors $\overline{z} = \overline{(5+2i)^2} = \overline{(5+2i)}^2 = (5-2i)^2 = 5^2-2\times5\times2i+(2i)^2 = 25-20i+4i^2 = 25-20i+4\times(-1)$

$\overline{z} = 25-4-20i = \boxed{21-20i}$.

4) $z = \dfrac{i}{i+1}$. Alors $\overline{z} = \dfrac{\overline{i}}{\overline{i+1}} = \dfrac{-i}{-i+1} = \dfrac{-i}{1-i} = \dfrac{-i(1+i)}{1^2+1^2} = \dfrac{-i-i^2}{1+1} = \dfrac{-i-(-1)}{2} = \dfrac{-i+1}{2} = \boxed{\dfrac{1-i}{2}}$.

5) $z = \dfrac{2-3i}{5-i}$. Alors $\overline{z} = \dfrac{\overline{2-3i}}{\overline{5-i}} = \dfrac{2+3i}{5+i} = \dfrac{(2+3i)(5-i)}{5^2+1^2} = \dfrac{10-2i+15i-3i^2}{25+1} = \dfrac{10+13i-3\times(-1)}{26} = \dfrac{10+3+13i}{26}$

$\overline{z} = \dfrac{13+13i}{26} = \dfrac{\cancel{13}(1+i)}{\cancel{13}\times2} = \boxed{\dfrac{1+i}{2}}$.

Exercice 3

1) $Z = 2-3z$. Alors $\boxed{\overline{Z} = 2-3\overline{z}}$.

2) $Z = (1-iz)(z-i)$. Alors $\boxed{\overline{Z} = (1+i\overline{z})(\overline{z}+i)}$.

3) $Z = z^3+2z^2+3iz-4$. Alors $\boxed{\overline{Z} = \overline{z}^3+2\overline{z}^2-3i\overline{z}-4}$.

4) $Z = \dfrac{1+2iz}{3+z}$. Alors $\boxed{\overline{Z} = \dfrac{1-2i\overline{z}}{3+\overline{z}}}$.

✎ Exercice 4

$\boxed{Z = z^2 + 3z + 4}$.

Z est un réel $\iff \overline{Z} = Z \iff \overline{z}^2 + 3\overline{z} + 4 = z^2 + 3z + 4 \iff \overline{z}^2 + 3\overline{z} + \cancel{4} - z^2 - 3z - \cancel{4} = 0$

$\iff \overline{z}^2 - z^2 + 3\overline{z} - 3z = 0 \iff (\overline{z} + z)(\overline{z} - z) - 3(\overline{z} - z) = 0 \iff (\overline{z} - z)(\overline{z} + z - 3) = 0$

$\iff \overline{z} - z = 0$ ou $\overline{z} + z - 3 = 0$ (produit nul) $\iff \overline{z} = z$ ou $\overline{z} + z - 3 = 0$.

Or $\overline{z} = z \iff$ $\boxed{z \text{ est un réel}}$.

De plus, $\overline{z} + z - 3 = 0 \iff \overline{z} + z = 3 \iff 2\mathscr{R}e(z) = 3 \iff \boxed{\mathscr{R}e(z) = \dfrac{3}{2}}$.

$\boxed{\text{L'ensemble cherché est donc constitué de l'axe des réels (l'axe des abscisses) et de la droite d'équation } x = \dfrac{3}{2}}$.

✎ Exercice 5

$\boxed{Z = \dfrac{5z - 2}{z - 1} \text{ pour } z \neq 1}$.

Z est un imaginaire pur $\iff \overline{Z} = -Z \iff \dfrac{5\overline{z} - 2}{\overline{z} - 1} = -\dfrac{5z - 2}{z - 1} \iff \dfrac{5\overline{z} - 2}{\overline{z} - 1} = \dfrac{2 - 5z}{z - 1}$

$\iff (5\overline{z} - 2)(z - 1) = (\overline{z} - 1)(2 - 5z) \iff 5z\overline{z} - 5\overline{z} - 2z + 2 = 2\overline{z} - 5z\overline{z} - 2 + 5z$

$\iff 5z\overline{z} - 5\overline{z} - 2z + 2 - 2\overline{z} + 5z\overline{z} + 2 - 5z = 0 \iff 10z\overline{z} - 7\overline{z} - 7z + 4 = 0 \iff 10|z|^2 - 7(z + \overline{z}) + 4 = 0$

$\iff 10|z|^2 - 7 \times 2\mathscr{R}e(z) + 4 = 0 \iff 5|z|^2 - 7\mathscr{R}e(z) + 2 = 0 \iff |z|^2 - \dfrac{7}{5}\mathscr{R}e(z) = -\dfrac{2}{5}$

En posant $z = x + iy$, ceci équivaut à :

$x^2 + y^2 - \dfrac{7}{5}x = -\dfrac{2}{5} \iff x^2 - \dfrac{7}{5}x + y^2 = -\dfrac{2}{5} \iff \left(x - \dfrac{7}{10}\right)^2 - \dfrac{49}{100} + y^2 = -\dfrac{2}{5} \iff \left(x - \dfrac{7}{10}\right)^2 + y^2 = -\dfrac{2}{5} + \dfrac{49}{100}$

$\iff \left(x - \dfrac{7}{10}\right)^2 + (y - 0)^2 = -\dfrac{40}{100} + \dfrac{49}{100} \iff \left(x - \dfrac{7}{10}\right)^2 + (y - 0)^2 = \dfrac{9}{100} \iff \left(x - \dfrac{7}{10}\right)^2 + (y - 0)^2 = \left(\dfrac{3}{10}\right)^2$.

$\boxed{\text{L'ensemble cherché est donc le cercle de centre } I\left(\dfrac{7}{10}\ ;\ 0\right) \text{ et de rayon } \dfrac{3}{10}\text{, privé du point } A(1\ ;\ 0)}$.